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ABSTRACT

Convexity preserving properties of certain totally positive density functions are shown to hold under weaker restrictions. These results generalize work of Karlin (1963) and Karlin and Proschan (1960) concerning convexity preserving transformations.

ON CONVEXITY PRESERVING FAMILIES
OF PROBABILITY DISTRIBUTIONS[†]

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W. R. van Zwet[††]

1. INTRODUCTION

Let F_θ be a family of probability distribution functions on R^1 with parameter $\theta \in T \subseteq R^1$, and let X denote the union of the supports of these distributions. For $k \geq 0$, let $\{g_0, g_1, \ldots, g_{k+1}\}$ be a set of real-valued finite functions on X that are integrable with respect to F_θ for all $\theta \in T$ and define

$$x_i(\theta) = \int g_i(x)\, dF_\theta(x), \quad i = 0, 1, \ldots, k+1 . \tag{1.1}$$

Following S. Karlin and W. J. Studden in [3] with a minor modification, we shall say that $\{g_0, g_1, \ldots, g_{k+1}\}$ constitute a *weak complete Tchebycheff system* (WCT-system) if for each $0 \leq m \leq k+1$ and all $x_0 < x_1 < \ldots < x_m \in X$ the determinant

$$\det(g_i(x_j))_{i,j=0,\ldots,m} \geq 0 ; \tag{1.2}$$

the system is called a *complete Tchebycheff system* (CT-system) if the inequality is always strict. The difference between this definition of a WCT-system and the one given in [3] is that we retain the case where $g_0, g_1, \ldots, g_m$ are linearly dependent on X for some $m \leq k+1$; in that case any choice of $g_{m+1}, \ldots, g_{k+1}$ will trivially satisfy definition (1.2). We shall also express inequalities

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(1.2) by saying that g_{k+1} is *generalized convex* with respect to the WCT-system $\{g_0, \ldots, g_k\}$.

Our discussion of WCT-systems will involve the related concept of total positivity (cf. [1]). A function $f(x, \theta)$ on $X \times T$ is said to be *totally positive of order* $n(TP_n)$ if for every $1 \leqq m \leqq n$, all $x_1 < x_2 < \ldots < x_m \in X$ and all $\theta_1 < \theta_2 < \ldots < \theta_m \in T$,

$$\det(f(x_i, \theta_j))_{i,j=1, \ldots, m} \geqq 0 . \tag{1.3}$$

The first question that comes to mind in this context is whether one can find conditions on the family F_θ that ensure that $\{\chi_0, \chi_1, \ldots, \chi_{k+1}\}$ will be a WCT-system on T whenever $\{g_0, g_1, \ldots, g_{k+1}\}$ constitutes a WCT-system on X. If the family F_θ possesses densities $p(x, \theta)$ with respect to a σ-finite measure μ with spectrum X and hence

$$\chi_i(\theta) = \int g_i(x)\, p(x, \theta)\, d\mu(x) ,$$

this question is easily answered. We have for each $0 \leqq m \leqq k + 1$ (cf. [1])

$$\det(\chi_i(\theta_j)) = \underset{x_0<x_1<\ldots<x_m}{\int \cdots\cdots\cdots \int} \det(g_i(x_j)) \det(p(x_i, \theta_j))\, d\mu(x_0) \ldots d\mu(x_m)$$

where in each determinant i and j run from 0 to m. It follows that the condition that p is TP_{k+2} is certainly sufficient; since we require that $\{\chi_0, \ldots, \chi_{k+1}\}$ will inherit the WCT-property for every WCT-system $\{g_0, \ldots, g_{k+1}\}$, the condition is essentially also necessary (by "essentially" is meant that for any $\theta_1 < \ldots < \theta_m$ the defining inequality (1.3) need not hold on a set of product-measure 0). We note that the fact that F_θ are probability distribution functions is not used in establishing the condition.

In view of this general result it is hardly surprising that recent discussions of convexity preserving properties (cf. [1] and [2]) have been confined to families of densities that are totally positive of the appropriate order. However, one usually does not discuss the class of all WCT-systems of a given order but restricts attention to a relatively small subclass (e.g. the case where $g_i = g_1^i$ for $i = 0, 1, \ldots, k$) . Also one often imposes additional restrictions on the family F_θ to ensure that for those systems $\{g_0, \ldots, g_{k+1}\}$ that are considered, $\{\chi_0, \ldots, \chi_{k+1}\}$ will also belong to some restricted class.

In Sections 3 and 4 of this paper we investigate how far the TP_{k+2} condition for p can be relaxed in two such restricted cases that seem to be important in practice. Like Section 1, the second section is of an expository character.

2. CONVEXITY OF ORDER k

Let f be a real-valued finite function defined on an arbitrary set $Y \subseteq R^1$. For $k \geqq 0$ we shall say that f is *convex of order* k (C_k) if f is generalized convex with respect to the CT-system $\{1, y, y^2, \ldots, y^k\}$, i.e., if for all $y_1 < y_2 < \ldots < y_{k+2}$

$$D_f(y_1, \ldots, y_{k+2}) = \begin{vmatrix} 1 & y_1 & y_1^2 & \cdots & y_1^k & f(y_1) \\ 1 & y_2 & y_2^2 & \cdots & y_2^k & f(y_2) \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ 1 & y_{k+2} & y_{k+2}^2 & \cdots & y_{k+2}^k & f(y_{k+2}) \end{vmatrix} \geq 0 . \qquad (2.1)$$

For $k = 0, 1$, (2.1) reduces to the ordinary definitions of nondecreasing or (measurable) convex functions. Generally speaking (2.1) is an extension of the concept of nonnegative $(k + 1)$ - th derivative. C_k functions were extensively studied by T. Popoviciu in [6]. We note that S. Karlin [1] refers to C_k functions as convex of order $(k + 1)$.

If P_m denotes a polynomial of degree at most m, then equivalent definitions of the C_k property are obviously

(A) (cf. [1]). For every P_k, $f - P_k$ changes sign at most $(k + 1)$ times on Y. If it does have $(k + 1)$ sign-changes, the signs occur in the order $(-)^{k+1}, (-)^k, \ldots, +, -, +$ for increasing values of the argument.

(B) For every $y_1 < y_2 < \ldots < y_{k+2} \in Y$, the P_{k+1} having $P_{k+1}(y_i) = f(y_i)$, $i = 1, 2, \ldots, k + 2$, has nonnegative coefficient for its $(k + 1)$ - th degree term.

There is also a close connection with differences. Let

$$\Delta_h^1 f(y) = f(y + h) - f(y)$$
$$\Delta_h^m f(y) = \Delta_h^1 \Delta_h^{m-1} f(y) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(y + jh) \tag{2.2}$$

and generally

$$\Delta_{h_1,\ldots,h_m}^m f(y) = \Delta_{h_m}^1 \Delta_{h_1,\ldots,h_{m-1}}^{m-1} f(y) =$$
$$= \sum_{j=0}^{m} (-1)^{m-j} \sum_{1\le i_1<i_2<\ldots<i_j\le m} f\left(y + \sum_{\nu=1}^{j} h_{i_\nu}\right). \tag{2.3}$$

Furthermore let

$$D_f^*(y_1, \ldots, y_{k+2}) = \frac{D_f(y_1, \ldots, y_{k+2})}{\prod_{1\le i<j\le k+2} (y_j - y_i)} ; \tag{2.4}$$

since the denominator is positive for $y_1 < y_2 < \ldots < y_{k+2}$, D_f may be replaced by D_f^* in definition (1.1). The following relation between D_f^* and differences may be proved by induction on k .

Lemma 2.1:

If Π denotes the set of permutations $\pi = (\pi(1), \pi(2), \ldots, \pi(k + 1))$ of the numbers $1, 2, \ldots, k + 1$, then

$$\Delta_{h_1,\ldots,h_{k+1}}^{k+1} f(y) = \sum_{i=1}^{k+1} h_i \sum_{\pi\in\Pi} D_f^*\left(y, y + h_{\pi(1)}, \ldots, y + \sum_{\nu=1}^{k+1} h_{\pi(\nu)}\right) . \tag{2.5}$$

We note that for $h_1 = h_2 = \dots = h_{k+1} = h$, (2.5) reduces to

$$\Delta_h^{k+1} f(y) = (k+1)!\, h^{k+1} D_f^*(y, y+h, \dots, y+(k+1)h) . \tag{2.6}$$

It follows from Lemma 2.1 that if f is C_k on Y , then for all $h_1, h_2, \dots, h_{k+1} > 0$,

$$\Delta_{h_1,\dots,h_{k+1}}^{k+1} f(y) \geqq 0 \tag{2.7}$$

whenever defined, i.e., whenever all $y + \sum_{\nu=1}^{j} h_{i_\nu} \in Y$.

In the special case that Y is an interval there is also a converse result and the following definition of the C_k property is equivalent to (2.1) in this case:

(C) f is (Lebesgue)-measurable and for $h > 0$, $y \in Y$, $y + (k+1)h \in Y$,

$$\Delta_h^{k+1} f(y) \geqq 0 . \tag{2.8}$$

In this case, however, the C_k property is hardly a generalization of nonnegative $(k+1)$ - th derivative at all. In fact, if Y is an open interval and $k \geqq 1$, definition (2.1) ensures continuity of f on Y and is equivalent to

(D) f is $(k-1)$ times continuously differentiable and $f^{(k-1)}$ is convex on Y .

Finally we consider the special case where Y is a set of consecutive integers. For integer $h > 0$

$$\Delta_h^{k+1} f(y) = \sum_{h_1=0}^{h-1} \dots \sum_{h_{k+1}=0}^{h-1} \Delta_1^{k+1} f\left(y + \sum_{j=1}^{k+1} h_j\right) . \tag{2.9}$$

Combining (2.6) and (2.9) we find that the C_k property may be defined in this case by

(E) For all $y, y+k+1 \in Y$

$$\Delta_1^{k+1} f(y) \geqq 0 . \tag{2.10}$$

For further details concerning the definitions given above the reader is referred to [6].

Let f_1 and f_2 be real-valued finite functions on Y. We shall say that f_2 is C_k with respect to f_1 on Y if there exists a C_k function f on $f_1(Y)$ such that $f_2 = f(f_1)$ on Y. If f_1 is nondecreasing on Y and f_2 is constant on any set where f_1 is constant, this reduces to

$$\begin{vmatrix} 1 & f_1(y_1) & f_1^2(y_1) & \cdots & f_1^k(y_1) & f_2(y_1) \\ 1 & f_1(y_2) & f_1^2(y_2) & \cdots & f_1^k(y_2) & f_2(y_2) \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 1 & f_1(y_{k+2}) & f_1^2(y_{k+2}) & \cdots & f_1^k(y_{k+2}) & f_2(y_{k+2}) \end{vmatrix} \geqq 0 \tag{2.11}$$

for all $y_1 < y_2 < \cdots < y_{k+2} \in Y$.

3. PRESERVING CONVEXITY OF ORDER k

Returning to the setup of Section 1, we let g be a real-valued finite function on X that is integrable with respect to F_θ for all $\theta \in T$ and define

$$\chi(\theta) = \int g(x)\, dF_\theta(x) \ .$$

We shall say that the family F_θ preserves convexity of order k if χ is C_k on T whenever g is C_k on X, i.e., whenever g is generalized convex with respect to $\{1, x, \ldots, x^k\}$ then χ is generalized convex with respect to $\{1, \theta, \ldots, \theta^k\}$. In [1] S. Karlin has shown that if densities $p(x, \theta)$ with respect to μ exist, then a sufficient condition for F_θ to preserve convexity of order k is that p is TP_{k+2} and that whenever g is a polynomial of exact degree $m \leq k$, then χ is also a polynomial of exact degree m. According to the result of Section 1 the first part of this condition ensures that χ is generalized convex with respect to the WCT-system

$$\int x^i\, dF_\theta(x), \quad i = 0, 1, \ldots, k \ ,$$

whereas the second part ensures that this is equivalent to generalized convexity with respect to $\{1, \theta, \ldots, \theta^k\}$.

However, this condition is not necessary. For $k = 0$ a condition that is necessary as well as sufficient was given by J. Krzyz in [4].

<u>Lemma 3.1</u>:

χ is nondecreasing on T whenever g is nondecreasing on X if and only if the family F_θ is stochastically increasing (i.e., $F_\theta(x)$ is nonincreasing in θ for every fixed x).

Since the TP_2 property of p is equivalent to monotone likelihood ratio, Krzyz's condition is weaker than Karlin's for $k = 0$ (cf. [5]).

For general k it is also easy to find a necessary and sufficient condition, provided that we restrict attention to those C_k functions g that can be extended to a C_k function on an open interval containing X. Since the convex functions constitute a convex cone spanned by the linear functions and functions of the form

$$h(x) = \begin{cases} 0 & \text{for} \quad x \leq x_0 \\ x - x_0 & \text{for} \quad x > x_0 , \end{cases}$$

we find from definition D of Section 2 that the convex cone of C_k functions is spanned by the polynomials P_k of degree at most k and functions of the form

$$h_k(x) = \begin{cases} 0 & \text{for} \quad x \leq x_0 \\ (x - x_0)^k & \text{for} \quad x > x_0 . \end{cases}$$

For $k = 0$ this is obviously also true. It follows that it is sufficient as well as necessary to require that χ be C_k whenever g is of one of the forms mentioned above. However, if g is a polynomial of degree at most k, then so is $-g$ and as a result both χ and $-\chi$ are required to be C_k, which implies that χ is also a polynomial of degree at most k. Hence we have proved

<u>Lemma 3.2</u>:

χ is C_k on T whenever g is C_k on an open interval containing X, if and only if for every x_0

$$\int (x - x_0)^k \, dF_\theta(x)$$

is C_k on T and whenever g is a polynomial of degree at most k the same holds for χ.

We note that for $k \leq 1$ the condition that the C_k function g can be extended to a C_k function on an open interval containing X is always satisfied. For $k = 0$ the lemma reduces to Lemma 3.1.

Although for $k \geq 1$ Lemma 3.2 seems to be fairly useless for practical purposes, the results obtained so far do seem to indicate that there exists a large class of C_k preserving families that do not possess any total positivity properties. The results in the remainder of this section exhibit a number of these families.

Theorem 3.1:

Let F_0 and F be distribution functions with characteristic functions ϕ_0 and ϕ respectively, and suppose that F is infinitely divisible and has $F(-0) = 0$. If for $t \geq 0$, F_t denotes the distribution function corresponding to $\phi_0 \cdot \phi^t$, then the family F_t, $0 \leq t < \infty$, preserves convexity of all orders.

Proof:

Let G_t denote the distribution function corresponding to ϕ^t and let X_t, $t \geq 0$, be a stochastic process with nonnegative stationary independent increments for which X_0, $X_{s+t} - X_s$ and X_t, $s, t \geq 0$, have distribution functions F_0, G_t and F_t respectively. For fixed $t \geq 0$ and $h > 0$ define

$$Z_i = X_{t+ih} - X_{t+(i-1)h}, \quad i = 1, 2, \ldots, k+1 .$$

$Z_1, Z_2, \ldots, Z_{k+1}$ are independent and identically distributed random variables that are also independent of X_t. Hence, because of the exchangeability of $Z_1, \ldots, Z_{k+1}$,

$$E\left[\sum_{j=0}^{k+1} (-1)^{k+1-j}\binom{k+1}{j} g(X_{t+jh}) \mid X_t = x\right] =$$

$$= E\left[\sum_{j=0}^{k+1} (-1)^{k+1-j}\binom{k+1}{j} g(x + Z_1 + \ldots + Z_j)\right] =$$

$$= E\left[\sum_{j=0}^{k+1} (-1)^{k+1-j} \sum_{1 \leq i_1 < \ldots < i_j \leq k+1} g(x + Z_{i_1} + \ldots + Z_{i_j})\right] =$$

$$= E\left[\Delta^{k+1}_{Z_1,\ldots,Z_{k+1}} g(x)\right] .$$

Since $Z_1, \ldots, Z_{k+1} \geq 0$ with probability 1, the last expression is nonnegative for every C_k function g and all x by (2.7). As a result

$$\Delta^{k+1}_h \chi(t) = E\left[\sum_{j=0}^{k+1} (-1)^{k+1-j}\binom{k+1}{j} g(X_{t+jh})\right] \geq 0$$

for all $t \geq 0$ and $h > 0$. As χ is a measurable function defined on the interval $[0,\infty)$, it is C_k by definition C of Section 2.

If we consider only integer values of t in Theorem 3.1, we may drop the assumption that F is infinitely divisible without affecting the proof. The C_k character of χ on the integers now follows from $\Delta_1^{k+1} \chi \geq 0$ by definition E of Section 2. Specializing to the case where F_0 is degenerate at 0 we obtain:

Corollary 3.1:

Every family F_n, $n = 1, 2, \ldots,$ of n-fold convolutions of a distribution function F_1 having $F_1(-0) = 0$ preserves convexity of every order.

We note that the fact that F_n preserves convexity of order k was proved by S. Karlin and F. Proschan in [2] under the additional assumption that F_1

possesses a density p that is a Pólya frequency density of order $k + 2$ (i.e., $p(x - y)$ is TP_{k+2} in x and y) .

Another special case of Theorem 3.1 is obtained by assuming F to be degenerate at 1, in which case the theorem reduces to

Every location parameter family $F_\theta(x) = G(x - \theta)$, $-\infty < \theta < \infty$, preserves convexity of every order.

This result is of course rather trivial. Without invoking Theorem 3.1, it follows at once from

$$\Delta_h^{k+1} \chi(\theta) = \Delta_h^{k+1} \int g(x + \theta)\, dG(x) = \int \Delta_h^{k+1} g(x + \theta)\, dG(x) .$$

In the same manner one easily verifies

Every scale parameter family $F_\theta(x) = G(x/\theta)$, $0 < \theta < \infty$, preserves convexity of every odd order. If moreover $G(-0) = 0$, then the family preserves convexity of all orders.

4. INVARIANT CONVEXITY PRESERVING FAMILIES

Let g_1, g_2, χ_1 and χ_2 be defined as in Section 1. We shall say that F_θ is invariant convexity preserving if, whenever g_1 is nondecreasing and g_2 is convex with respect to g_1 on X, then χ_1 is nondecreasing and χ_2 is convex with respect to χ_1 on T. In terms of WCT-systems we may express this property by requiring that for every WCT-system of the form $\{1, g_1, g_2\}$ the corresponding system $\{1, \chi_1, \chi_2\}$ is also a WCT-system.

In the first place this definition asserts that the family F_θ preserves the monotonicity of g_1 and hence by Lemma 1 the family is stochastically increasing; F_θ also preserves convexity (of order 1) provided that the parameter is subjected to a suitable nondecreasing transformation

$$\eta = \eta(\theta) = \int x \, dF_\theta(x) .$$

Moreover, this convexity preserving property is invariant under nondecreasing transformations g_1 of the random variable, the appropriate monotone transformation of θ then becoming χ_1. It is precisely because of this invariance that we do not require that F_θ be convexity preserving with respect to θ itself; i.e., that η be linear in θ. This property would be destroyed by nonlinear transformations g_1 anyway and would only result in fixing a possibly awkward parametrization.

From the general result of Section 1 it follows that F_θ is invariant convexity preserving if the density p is TP_3. The following theorem provides a necessary and sufficient condition.

Theorem 4.1:

Define $\bar{F}_\theta(x) = 1 - F_\theta(x)$. The family F_θ is invariant convexity preserving if and only if $\{1, \bar{F}_\theta(x_1), \bar{F}_\theta(x_2)\}$ is a WCT-system on T for every fixed pair $x_1 < x_2$.

Proof:

The condition asserts that for $x_1 < x_2$ and $\theta_0 < \theta_1 < \theta_2$,

$$\begin{vmatrix} 1 & \bar{F}_{\theta_0}(x_1) \\ 1 & \bar{F}_{\theta_1}(x_1) \end{vmatrix} \geqq 0, \quad \begin{vmatrix} 1 & \bar{F}_{\theta_0}(x_1) & \bar{F}_{\theta_0}(x_2) \\ 1 & \bar{F}_{\theta_1}(x_1) & \bar{F}_{\theta_1}(x_2) \\ 1 & \bar{F}_{\theta_2}(x_1) & \bar{F}_{\theta_2}(x_2) \end{vmatrix} \geqq 0 . \tag{4.1}$$

The first inequality means that F_θ is stochastically increasing and we have already remarked that this is necessary and sufficient for χ_1 to be nondecreasing whenever g_1 is. We may therefore assume that $\bar{F}_\theta(x)$ is nondecreasing in θ for every fixed x and restrict attention to the second inequality.

Let g_1 be nondecreasing and let $g_2 = f(g_1)$ where f is convex on $g_1(X)$. Since a convex function can be extended to a convex function on an interval, the same reasoning that we used in the proof of Lemma 3.2 shows that we need only be concerned with functions f that are linear and functions f of the form

$$\begin{aligned} f(y) &= 0 && \text{for} \quad y \leqq y_0 \\ &y - y_0 && \text{for} \quad y > y_0 . \end{aligned} \tag{4.2}$$

Without loss of generality we may assume that $y_0 = g_1(x_0) \in g_1(X)$. For linear f , χ_2 is linear and hence convex with respect to χ_1 . Only functions f of the form (4.2) remain to be considered and as a result we have the following necessary and sufficient condition for a stochastically increasing family F_θ to be invariant convexity preserving:

For every nondecreasing g_1 and every $x_0 \varepsilon X$,

$$\chi_2(\theta) = \int_{x_0}^{\infty} (g_1(x) - g_1(x_0))\, dF_\theta(x)$$

is convex with respect to $\chi_1(\theta)$.

By an approximation argument one shows that it is sufficient to consider only those functions g_1 that are left-continuous, nondecreasing step-functions assuming finitely many values. But then the above condition becomes:

For all $m = 1, 2, \ldots,$ all $x_1 < x_2 < \ldots < x_m$, all $\alpha_i > 0$, $i = 1, 2, \ldots, m$, all $1 \leqq i_0 \leqq m$ and all c ,

$$\sum_{i=i_0}^{m} \alpha_i \bar{F}_\theta(x_i) \tag{4.3}$$

is convex with respect to

$$\sum_{i=1}^{m} \alpha_i \bar{F}_\theta(x_i) + c . \tag{4.4}$$

Since (4.4) is nondecreasing in θ and (4.3) is constant on any set where (4.4) is constant, the determinantal convexity definition (2.11) for $k = 1$ applies. By subtracting from the second column in this determinant we find that convexity of (4.3) with respect to (4.4) is equivalent to

$$\begin{vmatrix} 1 & \sum_{i=1}^{i_0-1} \alpha_i \bar{F}_{\theta_0}(x_i) & \sum_{i=i_0}^{m} \alpha_i \bar{F}_{\theta_0}(x_i) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & \sum_{i=1}^{i_0-1} \alpha_i \bar{F}_{\theta_2}(x_i) & \sum_{i=i_0}^{m} \alpha_i \bar{F}_{\theta_2}(x_i) \end{vmatrix} \tag{4.5}$$

$$= \sum_{i=1}^{i_0-1} \sum_{j=i_0}^{m} \alpha_i \alpha_j \begin{vmatrix} 1 & \bar{F}_{\theta_0}(x_i) & \bar{F}_{\theta_0}(x_j) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & \bar{F}_{\theta_2}(x_i) & \bar{F}_{\theta_2}(x_j) \end{vmatrix} \geqq 0 .$$

By choosing $i_0 = m = 2$ we find that condition (4.1) is necessary; since every term in (4.5) has $x_i < x_j$ it is also sufficient. This completes the proof of the theorem.

It may be of interest to compare the sufficient condition that F_θ possesses a TP_3 density $p(x, \theta)$ with the necessary and sufficient condition of the theorem. One easily shows that the TP_3 assumption for p implies that $\bar{F}_\theta(x)$ is TP_3 , or

$$\begin{vmatrix} \bar{F}_{\theta_0}(x_0) & \bar{F}_{\theta_0}(x_1) \\ \bar{F}_{\theta_1}(x_0) & \bar{F}_{\theta_1}(x_1) \end{vmatrix} \geqq 0, \quad \begin{vmatrix} \bar{F}_{\theta_0}(x_0) & \bar{F}_{\theta_0}(x_1) & \bar{F}_{\theta_0}(x_2) \\ \bar{F}_{\theta_1}(x_0) & \bar{F}_{\theta_1}(x_1) & \bar{F}_{\theta_1}(x_2) \\ \bar{F}_{\theta_2}(x_0) & \bar{F}_{\theta_2}(x_1) & \bar{F}_{\theta_2}(x_2) \end{vmatrix} \geqq 0 , \qquad (4.6)$$

for $x_0 < x_1 < x_2$ and $\theta_0 < \theta_1 < \theta_2$. By letting x_0 tend to $-\infty$ we see that (4.6) implies (4.1). Hence the condition that $\bar{F}_\theta(x)$ be TP_3 is also sufficient for F_θ to be invariant convexity preserving.

If we restrict ourselves to the special case where the parameter set T is an interval and $\bar{F}_\theta(x)$ is differentiable with respect to θ , it turns out that theorem 4.1 involves a TP_2 instead of a TP_3 condition.

Theorem 4.2:

Let T be an interval and let $q(x, \theta) = \frac{\partial}{\partial\theta} \bar{F}_\theta(x)$ be defined on T for all x . Then the family F_θ is invariant convexity preserving if and only if q is TP_2 .

<u>Proof</u>:

The first inequality in (4.1) is equivalent to $q \geq 0$. Since $\bar{F}_\theta(x_2)$ is constant on any set where $\bar{F}_\theta(x_1) + \bar{F}_\theta(x_2)$ is constant and the latter is nondecreasing in θ , the second inequality of (4.1) asserts that $\bar{F}_\theta(x_2)$ is convex with respect to $\bar{F}_\theta(x_1) + \bar{F}_\theta(x_2)$. This in turn is equivalent to $q(x_1, \theta_1)\, q(x_2, \theta_2) - q(x_1, \theta_2)\, q(x_2, \theta_1) \geq 0$ for $x_1 < x_2$ and $\theta_1 < \theta_2$.

It is tempting to ask whether Theorem 4.2 can be generalized. One conceivable generalization would deal with invariant C_k preserving families F_θ , i.e., families for which χ_1 is nondecreasing and χ_2 is C_k with respect to χ_1 whenever g_1 is nondecreasing and g_2 is C_k with respect to g_1 . However, even a cursory inspection shows that only trivial examples of such families exist. The necessary requirement that χ_2 be a polynomial in χ_1 of degree at most k whenever g_2 is a polynomial in g_1 of degree at most k , cannot be satisfied for every nondecreasing g_1 except in a trivial manner.

A more promising generalization is to consider families F_θ that transform WCT-systems $\{1, g_1, \ldots, g_{k+1}\}$ into WCT-systems $\{1, \chi_1, \ldots, \chi_{k+1}\}$. If one restrict attention to the case where X and T are intervals and g_1 and F_θ satisfy certain regularity conditions, one shows in a fairly straightforward manner that a necessary and sufficient condition on F_θ is that q be TP_{k+1} , thus generalizing Lemma 3.1 and Theorem 4.2 to the case where $k \geq 2$. We may conclude that although something may be lost for $k \geq 2$, the basic reason that Theorems 4.1 and 4.2 work is not the fact that $k = 1$ in that case, but that $g_0 \equiv 1$ and that F_θ are probability distribution functions.

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Convexity preserving properties of certain totally positive density functions are shown to hold under weaker restrictions. These results generalize work of Karlin (1963) and Karlin and Proschan (1960) concerning convexity preserving transformations.

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